

Renormalization for singular-potential scattering[†]

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ABSTRACT: In the calculation of quantum-mechanical singular-potential scattering, one encounters divergence. We suggest three renormalization schemes, dimensional renormalization, analytic continuation approach, and minimal-subtraction scheme to remove the divergence.

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[†] This is an enlarged version of the corresponding content in Ref. [1].

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1 Introduction

Once a potential is a singular potential, one encounters divergences. In order to remove the divergence, we need a renormalization procedure. In this paper, we suggest three renormalization schemes, dimensional renormalization, analytic continuation approach, and minimal-subtraction scheme for removing the divergence in singular-potential scattering.

In order to illustrate the validity of dimensional renormalization, we consider the Born approximation.

To demonstrate the validity of renormalization, by taking the Lennard-Jones potential as an example, through three different approaches, we perform renormalization treatments. It will be shown that the results obtained by these three different approaches are the same.

In the following, we take the Lennard-Jones potential $V(r) = \eta \frac{6}{m-6} \left(\alpha \frac{1}{r^m} - \frac{\beta}{6} \frac{m}{r^6} \right)$ as an example to testify the validity of the dimensional renormalization through a comparison with other two renormalization treatments.

The Lennard-Jones potential with $m = 12$ reads

$$V(r) = \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right). \quad (1.1)$$

A divergence appears in the first-order Born approximation of s -wave scattering phase shift [2]:

$$\delta_0(k) = -\frac{\pi}{2} \int_0^\infty r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr), \quad (1.2)$$

where $J_\nu(z)$ is the Bessel function of the first kind. In the following, we remove this divergence through three renormalization treatments, dimensional renormalization, the analytic continuation approach, and the minimal-subtraction scheme, to show the validity.

The dimensional renormalization approach is introduced in section 1.1. The analytic continuation approach is introduced in section 1.2. A minimal-subtraction scheme is introduced in section 1.3. The conclusion is given in section 2.

1.1 Dimensional renormalization

Using the n -dimensional Born approximation [3], we have the first-order scattering phase shift,

$$\delta_l^{(n)}(k) = -\frac{\pi}{2} \int_0^\infty r dr V(r) J_{n/2+l-1}^2(kr). \quad (1.3)$$

The n -dimensional s -wave phase shift of the Lennard-Jones potential (1.1) can be obtained by performing the integral directly,

$$\begin{aligned} \delta_0^{(n)}(k) &= -\frac{\pi}{2} \int_0^\infty r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{n/2-1}^2(kr) \\ &= -\frac{63\pi\alpha\eta k^{10} \Gamma(n/2-6)}{2^{10}\Gamma(n/2+5)} + \frac{3\pi\beta\eta k^4 \Gamma(n/2-3)}{2^4\Gamma(n/2+2)}. \end{aligned} \quad (1.4)$$

Then directly putting $n = 3$ gives

$$\delta_0^{(3)}(k) = \frac{2}{155925} \pi \alpha \eta k^{10} + \frac{2}{15} \pi \beta \eta k^4. \quad (1.5)$$

This is a finite result.

1.2 Analytic continuation approach

As comparison, we now use another renormalization treatment to remove the divergence, which is based on analytic continuation.

The integral $I = \int_0^\infty f(x) dx$ will diverge, if the expansion of $f(x)$ at $x = 0$ has negative-power terms. In order to remove the divergence, we use the analytic continuation technique. Concretely, we rewrite the integral as [4]

$$\begin{aligned} I &= \int_0^\infty f(x) dx \\ &= \int_0^1 \left(f(x) - \sum_{n=2}^N a_n \frac{1}{x^n} \right) dx + \sum_{n=2}^N a_n \frac{x^{1-n}}{1-n} + \int_1^\infty f(x) dx, \end{aligned} \quad (1.6)$$

where a_n is the expansion coefficient and N equals the highest negative power of the expansion of $f(x)$. Here the integral is split into two parts: $\int_1^\infty dx$ and $\int_0^1 dx$. The integral $\int_1^\infty dx$ is well defined. The divergence encountered in the integral $\int_0^1 dx$ is removed by $\sum_{n=2}^N a_n \frac{1}{x^n}$. The basis of eq. (1.6) is essentially analytic continuation.

First, split the integral in Eq. (1.2) into two parts:

$$\delta_0(k) = [\delta_0(k)]_0^\epsilon + [\delta_0(k)]_\epsilon^\infty, \quad (1.7)$$

where

$$[\delta_0(k)]_0^\epsilon = -\frac{\pi}{2} \int_0^\epsilon r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr), \quad (1.8)$$

$$[\delta_0(k)]_\epsilon^\infty = -\frac{\pi}{2} \int_\epsilon^\infty r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr), \quad (1.9)$$

where ϵ is a finite number.

The integral in Eq. (1.9) can be performed directly,

$$\begin{aligned}
[\delta_0(k)]_\epsilon^\infty = & -\frac{\alpha\eta}{22k\epsilon^{11}} + \frac{\beta\eta}{5k\epsilon^5} - \text{Si}(2k\epsilon) \left(\frac{4\alpha\eta k^{10}}{155925} + \frac{4\beta\eta k^4}{15} \right) + \frac{2\pi\alpha\eta k^{10}}{155925} + \frac{2\pi\beta\eta k^4}{15} \\
& + \sin(2k\epsilon) \left[-\frac{\alpha\eta}{110\epsilon^{10}} + \frac{\alpha\eta k^2}{1980\epsilon^8} - \frac{\alpha\eta k^4}{20790\epsilon^6} + \left(\frac{\alpha\eta k^6}{103950} + \frac{\beta\eta}{10} \right) \frac{1}{\epsilon^4} - \left(\frac{\alpha\eta k^8}{155925} - \frac{\beta\eta k^2}{15} \right) \frac{1}{\epsilon^2} \right] \\
& + \cos(2k\epsilon) \left[\frac{\alpha\eta}{22k\epsilon^{11}} - \frac{\alpha\eta k}{495\epsilon^9} + \frac{\alpha\eta k^3}{6930\epsilon^7} - \left(\frac{\alpha\eta k^5}{51975} + \frac{\beta\eta}{5k} \right) \frac{1}{\epsilon^5} \right. \\
& \left. + \left(\frac{\alpha\eta k^7}{155925} + \frac{\beta\eta k}{15} \right) \frac{1}{\epsilon^3} - \left(\frac{2\alpha\eta k^9}{155925} - \frac{2\beta\eta k^3}{15} \right) \frac{1}{\epsilon} \right], \tag{1.10}
\end{aligned}$$

where $\text{Si}(z)$ is the sine integral [5].

Now we deal with $[\delta_0(k)]_0^\epsilon$.

Expanding the integrand in $[\delta_0(k)]_0^\epsilon$ around $r = 0$ gives

$$r\eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr) = D(r), \tag{1.11}$$

where

$$D(r) = \frac{2\alpha\eta k}{\pi r^{10}} - \frac{2\alpha\eta k^3}{3\pi r^8} + \frac{4\alpha\eta k^5}{45\pi r^6} - \left(\frac{2\alpha\eta k^7}{315\pi} + \frac{4\beta\eta k}{\pi} \right) \frac{1}{r^4} + \left(\frac{4\alpha\eta k^9}{14175\pi} + \frac{4\beta\eta k^3}{3\pi} \right) \frac{1}{r^2} + \dots \tag{1.12}$$

According to Eq. (1.6), subtracting $D(r)$ from the integral in Eq. (1.8) gives

$$\begin{aligned}
& -\frac{\pi}{2} \int_0^\epsilon dr \left[r\eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr) - D(r) \right] \\
& = \frac{\alpha\eta}{22k\epsilon^{11}} - \frac{\alpha\eta k}{9\epsilon^9} + \frac{\alpha\eta k^3}{21\epsilon^7} - \left(\frac{\beta\eta}{5k} + \frac{2\alpha\eta k^5}{225} \right) \frac{1}{\epsilon^5} + \left(\frac{\alpha\eta k^7}{945} + \frac{2\beta\eta k}{3} \right) \frac{1}{\epsilon^3} - \left(\frac{2\alpha\eta k^9}{14175} + \frac{2\beta\eta k^3}{3} \right) \frac{1}{\epsilon} \\
& + 4 \text{Si}(2k\epsilon) \left(\frac{\alpha\eta k^{10}}{155925} + \frac{\beta\eta k^4}{15} \right) \\
& + \sin(2k\epsilon) \left[\frac{\alpha\eta}{110\epsilon^{10}} - \frac{\alpha\eta k^2}{1980\epsilon^8} + \frac{\alpha\eta k^4}{20790\epsilon^6} - \left(\frac{\alpha\eta k^6}{103950} + \frac{\beta\eta}{10} \right) \frac{1}{\epsilon^4} + \left(\frac{\alpha\eta k^8}{155925} + \frac{\beta\eta k^2}{15} \right) \frac{1}{\epsilon^2} \right] \\
& + \cos(2k\epsilon) \left[-\frac{\alpha\eta}{22k\epsilon^{11}} + \frac{\alpha\eta k}{495\epsilon^9} - \frac{\alpha\eta k^3}{6930\epsilon^7} + \left(\frac{\alpha\eta k^5}{51975} + \frac{\beta\eta}{5k} \right) \frac{1}{\epsilon^5} \right. \\
& \left. - \left(\frac{\alpha\eta k^7}{155925} + \frac{\beta\eta k}{15} \right) \frac{1}{\epsilon^3} + \left(\frac{2\alpha\eta k^9}{155925} + \frac{2\beta\eta k^3}{15} \right) \frac{1}{\epsilon} \right]. \tag{1.13}
\end{aligned}$$

Introducing

$$D_s(r) = \frac{2\alpha\eta k}{\pi r^{10s}} - \frac{2\alpha\eta k^3}{3\pi r^{8s}} + \frac{4\alpha\eta k^5}{45\pi r^{6s}} - \left(\frac{2\alpha\eta k^7}{315\pi} + \frac{4\beta\eta k}{\pi} \right) \frac{1}{r^{4s}} + \left(\frac{4\alpha\eta k^9}{14175\pi} + \frac{4\beta\eta k^3}{3\pi} \right) \frac{1}{r^{2s}} + \dots \tag{1.14}$$

with $D_s(r)|_{s=1} = D(r)$.

Integrating $D_s(r)$ and then putting $s = 1$ give

$$\begin{aligned}
& -\frac{\pi}{2} \int_0^\epsilon dr D_s(r) \Big|_{s=1} \\
&= \frac{\eta k}{14175} \left[\frac{14175\alpha}{(10s-1)\epsilon^{10s-1}} - \frac{4725\alpha k^2}{(8s-1)\epsilon^{8s-1}} + \frac{630\alpha k^4}{(6s-1)\epsilon^{6s-1}} - \frac{45(630\beta + \alpha k^6)}{(4s-1)\epsilon^{4s-1}} + \frac{2k^2(4725\beta + \alpha k^6)}{(2s-1)\epsilon^{2s-1}} \right] \Big|_{s=1} \\
&= \frac{\alpha\eta k}{9\epsilon^9} - \frac{\alpha\eta k^3}{21\epsilon^7} + \frac{2\alpha\eta k^5}{225\epsilon^5} - \left(\frac{\alpha\eta k^7}{945} + \frac{2\beta\eta k}{3} \right) \frac{1}{\epsilon^3} + \left(\frac{2\beta\eta k^3}{3} + \frac{2\alpha\eta k^9}{14175} \right) \frac{1}{\epsilon}. \tag{1.15}
\end{aligned}$$

Then we arrive at

$$\begin{aligned}
\delta_0(k) &= -\frac{\pi}{2} \int_0^\epsilon dr \left[r\eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr) - D(r) \right] + \left(-\frac{\pi}{2} \int_0^\epsilon dr D_s(r) \Big|_{s=1} \right) \\
&\quad - \frac{\pi}{2} \int_\epsilon^\infty r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr) \\
&= \frac{2}{155925} \pi \alpha \eta k^{10} + \frac{2}{15} \pi \beta \eta k^4. \tag{1.16}
\end{aligned}$$

This result agrees with that given by dimensional renormalization.

1.3 Minimal-subtraction scheme

The minimal-subtraction scheme is simply to remove the poles in divergent quantities [6].

First directly cut off the lower limit of the integral in Eq. (1.2):

$$\delta_0(k, \epsilon) = -\frac{\pi}{2} \int_\epsilon^\infty r dr \eta \left(\frac{\alpha}{r^{12}} - \frac{2\beta}{r^6} \right) J_{1/2}^2(kr). \tag{1.17}$$

For $\epsilon > 0$, the integral is convergent.

Performing the integral in Eq. (1.17) gives

$$\begin{aligned}
\delta_0(k, \epsilon) &= -\frac{\alpha\eta}{22k\epsilon^{11}} + \frac{\beta\eta}{5k\epsilon^5} + \frac{2\pi\alpha\eta k^{10}}{155925} + \frac{2\pi\beta\eta k^4}{15} - \text{Si}(2k\epsilon) \left(\frac{4\alpha\eta k^{10}}{155925} + \frac{4\beta\eta k^4}{15} \right) \\
&\quad + \cos(2k\epsilon) \left[\frac{\alpha\eta}{22k\epsilon^{11}} - \frac{\alpha\eta k}{495\epsilon^9} + \frac{\alpha\eta k^3}{6930\epsilon^7} - \frac{\alpha\eta k^5}{51975\epsilon^5} - \frac{\beta\eta}{5k\epsilon^5} \right. \\
&\quad \left. + \left(\frac{\alpha\eta k^7}{155925} + \frac{\beta\eta k}{15} \right) \frac{1}{\epsilon^3} - \left(\frac{2\alpha\eta k^9}{155925} + \frac{2\beta\eta k^3}{15} \right) \frac{1}{\epsilon} \right] \\
&\quad + \sin(2k\epsilon) \left[-\frac{\alpha\eta}{110\epsilon^{10}} + \frac{\alpha\eta k^2}{1980\epsilon^8} - \frac{\alpha\eta k^4}{20790\epsilon^6} + \left(\frac{\alpha\eta k^6}{103950} + \frac{\beta\eta}{10} \right) \frac{1}{\epsilon^4} - \left(\frac{\alpha\eta k^8}{155925} + \frac{\beta\eta k^2}{15} \right) \frac{1}{\epsilon^2} \right]. \tag{1.18}
\end{aligned}$$

Expanding $\delta_0(k, \epsilon)$ around $\epsilon = 0$ gives

$$\delta_0(k, \epsilon) = -\frac{\alpha\eta k}{9\epsilon^9} + \frac{\alpha\eta k^3}{21\epsilon^7} - \frac{2\alpha\eta k^5}{225\epsilon^5} + \frac{\alpha\eta k^7 + 630\beta\eta k}{945\epsilon^3} - \frac{2\alpha\eta k^9 + 9450\beta\eta k^3}{14175\epsilon} + \frac{2\pi\alpha\eta k^{10} + 20790\pi\beta\eta k^4}{155925} + O(\epsilon). \tag{1.19}$$

Take $\epsilon \rightarrow 0$ and, according to the minimal-subtraction scheme, dropping out the terms that diverge when $\epsilon \rightarrow 0$ give

$$\delta_0(k) = \frac{2}{155925} \pi \alpha \eta k^{10} + \frac{2}{15} \pi \beta \eta k^4. \tag{1.20}$$

2 Conclusions

The above three renormalized results agree with each other demonstrates the validity of these three renormalization schemes.

In dimensional renormalization scheme, we need an arbitrary dimensional theory in which the value of spatial dimension n appears as a renormalization parameter, like that in dimensional renormalization in quantum field theory [3, 7–13]. Moreover, the above schemes can also be examined through exact solutions, e.g. [14]. The renormalization treatment introduced in the present paper should be applied to more general scattering problems [15].

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